

# On Fully Split Lacunary Polynomials in Finite Fields

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## Abstract

We estimate the number of possible types degree patterns of  $k$ -lacunary polynomials of degree  $t < p$  which split completely modulo  $p$ . The result is based on a combination of a bound on the number of zeros of lacunary polynomials with some graph theory arguments.

## 1 Introduction

Zeros and factorisations of lacunary polynomials, that is, polynomials of high degree with relatively small number of non-zero coefficients, has always been a subject of active investigation, see [2, 4, 7, 8, 10] and references therein. We say that a polynomial  $f$  over a field  $\mathbb{K}$  is  $k$ -lacunary if it has at most  $k+1$

non-zero coefficients, including a non-zero constant term, that is, if  $f(0) \neq 0$  and

$$f(X) = a_0 + a_1 X^{t_1} + \dots + a_k X^{t_k} \in \mathbb{K}[X] \quad (1)$$

for some positive integers  $t_1 < \dots < t_k$ .

For example, a classical result of Descartes asserts that a  $k$ -lacunary polynomial  $f \in \mathbb{R}[X]$  may have at most  $2k$  real roots. Furthermore, Lenstra [8] has shown that for an algebraic number field  $\mathbb{K}$  of degree  $m$  over  $\mathbb{Q}$  and a  $k$ -lacunary polynomial  $f \in \mathbb{K}[X]$ , the product  $g$  of all irreducible divisors  $h \mid f$  of degree at most  $\deg h \leq d$  is of degree

$$\deg g = O\left(k^2 2^{md} md \log(2mdk)\right).$$

Schinzel [10] has obtained a series of statistical results about the number of  $k$ -lacunary irreducible polynomials with prescribed coefficients. In particular, by [10, Corollary 2], for any algebraic numbers  $a_0, \dots, a_k$  there are at most  $O\left(T^{\lfloor (k+1)/2 \rfloor}\right)$   $k$ -tuples of integers

$$\mathbf{t} = (t_1, \dots, t_k), \quad 1 \leq t_1 < \dots < t_k, \quad (2)$$

with  $t_k \leq T$  and such that the largest non-cyclotomic factor (that is, a factor which does not have roots that are roots of unity) of the  $k$ -lacunary polynomial (1) is reducible over  $\mathbb{K} = \mathbb{Q}(a_1/a_0, \dots, a_k/a_0)$ .

Here we consider a related question about estimating the number  $N_k(p, t)$  of  $k$ -tuples (2) such that there is a  $k$ -lacunary polynomial of the form (1) of degree  $t_k = t$  over the finite field  $\mathbb{K} = \mathbb{F}_p$  of  $p$  elements, where  $p$  is a prime, that fully splits over  $\mathbb{F}_p$ .

**Theorem 1.** *If a positive integer  $k$  is fixed then for any prime  $p$  and positive integer  $t < p$ , we have,*

$$N_k(p, t) \leq t^{k - k \lceil (k-3)/2 \rceil - 1} p^{(k-1) \lceil (k-3)/2 \rceil + o(1)}$$

as  $p \rightarrow \infty$ .

Clearly, Theorem 1 is nontrivial only for  $k > 3$  and for

$$t > p^{1-1/k+\varepsilon}, \quad (3)$$

with some fixed  $\varepsilon > 0$ . Furthermore, for  $t \gg p$  we obtain the bound

$$N_k(p, t) \leq t^{\lceil k/2 \rceil + 1 + o(1)}.$$

Our result is based on a rather unusual combination of two techniques: a bound on the number of zeros of lacunary polynomials (see Section 2) and a bound on the so-called domination number of a graph (see Section 3).

Throughout the paper, the implied constants in the symbols ‘ $O$ ’, ‘ $\ll$ ’ and ‘ $\gg$ ’ may depend on  $k$  (we recall that the notations  $U \ll V$  and  $V \gg U$  is equivalent to  $U = O(V)$ ).

## 2 Zeros of Lacunary Polynomials

We need the following estimate from [1] on the number of zeros of lacunary polynomials over  $\mathbb{F}_p$ .

**Lemma 2.** *For  $k+1 \geq 2$  elements  $a_0, a_1, \dots, a_k \in \mathbb{F}_p^*$  and integers  $0 = t_0 < t_1 < \dots < t_k < p$ , the number of solutions  $Q$  to the equation*

$$\sum_{i=0}^k a_i x^{t_i} = 0, \quad x \in \mathbb{F}_p^*,$$

*with  $t_0 = 0$ , satisfies*

$$Q \leq 2p^{1-1/k} D^{1/k} + O(p^{1-2/k} D^{2/k}),$$

*where*

$$D = \min_{0 \leq i \leq k} \max_{j \neq i} \gcd(t_j - t_i, p-1).$$

**Lemma 3.** *For  $k+1 \geq 2$  elements  $a_0, a_1, \dots, a_k \in \mathbb{F}_p^*$  and integers  $0 = t_0 < t_1 < \dots < t_k < p$ , the multiplicity of any root  $\rho$  of the polynomial*

$$\sum_{i=0}^k a_i X^{t_i} \in \mathbb{F}_p[X]$$

*is at most  $k$ .*

*Proof.* Let

$$F(X) = \sum_{i=0}^k a_i X^{t_i}.$$

Then for the  $j$  derivative  $F^{(j)}(X)$  we have

$$F^{(j)}(X)X^j = \sum_{i=0}^k \prod_{h=0}^{j-1} (t_i - h) a_i X^{t_i}$$

(where as usual, we set  $F^{(0)}(X) = F(X)$ ). Thus, if  $r \neq 0$  is a root of multiplicity at least  $k+1 \leq t_k < p$  in the algebraic closure of  $\mathbb{F}_p$ , then

$$F^{(j)}(r) = 0, \quad j = 0, \dots, k.$$

Therefore, the homogeneous system of equations

$$\sum_{i=0}^k \prod_{h=0}^{j-1} (t_i - h) x_i = 0, \quad j = 0, \dots, k,$$

has a non-zero solution  $x_i = a_i r^{t_i}$ ,  $i = 0, \dots, k$ . This implies

$$\det \left[ \left( \prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\dots,k} \right] = 0,$$

which is impossible for  $0 = t_0 < t_1 < \dots < t_k < p$  as an easy calculation shows that

$$\det \left[ \left( \prod_{h=0}^{j-1} (t_i - h) \right)_{i,j=0,\dots,k} \right] = \prod_{0 \leq i < j \leq k} (t_j - t_i) \neq 0.$$

The above contradiction implies the desired result.  $\square$

### 3 Domination Number of a Graph

Let  $G = (V, E)$  be a simple undirected graph of order  $n$ . A *dominating set*  $S$  of  $G$  is a vertex subset such that any vertex of  $V \setminus S$  has a neighbour in  $S$ . Intuitively, a dominating set of a graph is a vertex subset whose neighbours, along with themselves, make up the vertex set of the graph.

The minimum cardinality of a dominating set of  $G$  is called the *domination number*  $\gamma(G)$  of  $G$ . In other words,

$$\gamma(G) = \min_{S \subseteq V(G)} \left\{ |S| : V(G) \subseteq \bigcup_{v \in S} \hat{N}(v) \right\},$$

where  $\hat{N}(v)$  denotes the closed neighbourhood of a vertex  $v$ .

We denote by  $\delta(G)$  the minimum degree of  $G$ .

When  $\delta(G)$  is big enough, there are very good upper bounds for the domination number of the graph  $G$  in terms of  $\delta(G)$  and  $n$  (see, for example, [3, 6]). However, for small values of  $\delta(G)$  the classical result of Ore [9] is stronger and provides an upper bound for the domination number of a graph with no isolated vertices:

**Lemma 4.** *If  $G$  is a graph of order  $n$  with  $\delta(G) \geq 1$ , then*

$$\gamma(G) \leq \frac{n}{2}.$$

## 4 Proof of Theorem 1

Since  $p > t_k$ , by Lemma 3 the multiplicity of each non-zero root of a polynomial of the form (1) does not exceed  $k$ . Hence, if a polynomial  $F(X) \in \mathbb{F}_p[X]$  of the form (1) splits completely over  $\mathbb{F}_p$  then the equation

$$a_0 + a_1x^{t_1} + \dots + a_nx^{t_k} = 0, \quad x \in \mathbb{F}_p^*,$$

with  $1 \leq t_1 < \dots < t_k$  has at least  $t_k/k$  solutions. Then, from Lemma 2 we have

$$t_k/k = O\left(p^{1-1/k} D_{\mathbf{t}}^{1/k}\right),$$

where

$$D_{\mathbf{t}} = \min_{0 \leq i \leq n} \max_{j \neq i} \gcd(t_j - t_i, p-1).$$

Thus  $D_{\mathbf{t}}t \mid p-1$  and, since  $k$  is fixed,

$$t \geq D_{\mathbf{t}} \gg t_k^k p^{-(k-1)} = t^k p^{-(k-1)}. \quad (4)$$

We now fix  $D \mid p-1$ , and for each  $\mathbf{t} = (t_1, \dots, t_k)$  construct a graph  $G_{\mathbf{t}}(D)$  on  $k+1$  vertices  $0, \dots, k$ , connecting  $i$  and  $j$  if and only if  $\gcd(t_i - t_j, p-1) \geq D$  (where, as before  $t_0 = 0$ ).

Clearly, if  $D_{\mathbf{t}} = D$  and  $G_{\mathbf{t}}(D) = G$  then  $\delta(G) \geq 1$ .

Now, for a fixed positive integer  $D \leq t < p$  and a graph  $G$  with  $k+1$  vertices and  $\delta(G) \geq 1$ , we estimate the number  $M_p(D, G, t)$  of vectors  $\mathbf{t} =$

$(t_1, \dots, t_k) \in \mathbb{Z}^k$  with  $1 \leq t_1 < \dots < t_k$  and  $t_k = t$  such that  $G_{\mathbf{t}}(D) = G$ . Summing over all graphs  $G$  (since  $k$  is fixed there are only finitely many graphs) and admissible values of  $D$ , that is, with  $t \geq D \gg t^k p^{-(k-1)}$ , see (4), leads to the desired estimate.

Given a graph  $G$  with  $k+1$  vertices and  $\delta(G) \geq 1$ , we now fix a dominating set  $S$  in  $G$  of cardinality  $\#S = \lfloor (k+1)/2 \rfloor$ , which exists by Lemma 4 (obviously, we can always add more vertices to  $S$  if necessary to guarantee  $\#S = \lfloor (k+1)/2 \rfloor$ ). So for each  $j \notin S$  with  $j \neq 0, k$ , there is  $i \in S$  such that  $\gcd(t_i - t_j, q-1) \geq D$ . So if  $t_i$  is fixed, then  $t_j$  can take at most

$$\sum_{\substack{d|p-1 \\ d \geq D}} \frac{t}{d} \ll \frac{t}{D} \sum_{d|p-1} 1 = \frac{t}{D} p^{o(1)} \quad (5)$$

values, where we have used the known bound on the divisor function, (see [5, Theorem 320]). Finally, when  $t_k = t$  is fixed, each  $t_i$ ,  $i \in S$ , can take at most  $t$  values.

Furthermore, if both  $0, k \in S$  then there are only

$$\#S - 2 \leq \lfloor (k+1)/2 \rfloor - 2 = \lfloor (k-3)/2 \rfloor$$

elements  $t_i$  with  $i \in S \setminus \{0, k\}$  to be chosen. After all values of  $t_i$  with  $i \in S$  are fixed, we see from (5) that the remaining

$$k+1 - \#S = \lceil (k+1)/2 \rceil$$

elements  $t_j$ ,  $j \notin S$ , can be chosen in at most  $(tp^{o(1)}/D)^{\lceil (k+1)/2 \rceil}$  ways. So in this case

$$M_p(D, G, t) \leq t^{\lfloor (k-3)/2 \rfloor} (t/D)^{\lceil (k+1)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k+1)/2 \rceil} p^{o(1)}. \quad (6)$$

If  $0 \in S$  but  $k \notin S$ , or  $0 \notin S$  but  $k \in S$ , then the same argument implies:

$$M_p(D, G, t) \leq t^{\lfloor (k-1)/2 \rfloor} (t/D)^{\lceil (k-1)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k-1)/2 \rceil} p^{o(1)}. \quad (7)$$

Finally, if both  $0, k \notin S$  then we get

$$M_p(D, G, t) \leq t^{\lfloor (k+1)/2 \rfloor} (t/D)^{\lceil (k-3)/2 \rceil} p^{o(1)} = t^{k-1} D^{-\lceil (k-3)/2 \rceil} p^{o(1)}. \quad (8)$$

Clearly, bound (8) dominates the bounds (6) and (7). In particular, for  $t \geq D \gg t^k p^{-(k-1)}$  we obtain

$$M_p(D, G, t) \leq t^{k-1-k\lceil(k-3)/2\rceil} p^{(k-1)\lceil(k-3)/2\rceil+o(1)}.$$

Since, as we have mentioned, there are only finitely many possibilities for the graphs  $G_{\mathbf{t}}(D)$ , recalling (4) and the bound on the divisor function (see [5, Theorem 320]), we obtain the desired result.

## 5 Comments

A slight modification of our approach can easily produce a nontrivial bound for  $1 \leq k \leq 3$  as well, however we do not know how to relax the condition (3).

It is certainly an interesting question to show that almost all  $k$ -lacunary polynomials of a large degree are irreducible over  $\mathbb{F}_p$ . In fact, as a first step one can try to get a lower bound on the degree over  $\mathbb{F}_p$  of the splitting field of a “random”  $k$ -lacunary polynomial.

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